L' Hospital Rule

1. **(a)** $f(x) = \lim_{x\to 0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}$, $g(x) = \sin x$ $\quad\therefore\quad \lim_{x\to 0}\dfrac{f'(x)}{g'(x)} = \lim_{x\to 0}\dfrac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x}$

Since $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist, we cannot apply L'hospital rule.

(b) $f(x) = x - \sin x$, $g(x) = x + \sin x$ $\quad\therefore\quad \lim_{x\to\infty}\dfrac{f'(x)}{g'(x)} = \lim_{x\to\infty}\dfrac{1-\cos x}{1+\cos x}$

Since $\lim_{x\to\infty}\cos x$ does not exist, we cannot apply L'hospital rule.

(c) $f(x) = 2x + \sin 2x$, $g(x) = (2x\sin x)\,e^{\sin x}$ $\quad\therefore\quad \lim_{x\to\infty}\dfrac{f'(x)}{g'(x)} = \lim_{x\to\infty}\dfrac{1+\cos 2x}{e^{\sin x}(x\sin x\cos x + \sin x + x\cos x)}$

Since $\lim_{x\to\infty}(1+\cos 2x)$ does not exist, we cannot apply L'hospital rule.

2. **(a)** $$\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\cos 2x}\ \underset{(\text{LHR})}{(0/0\text{ form})} = \lim_{x\to\frac{\pi}{4}}\frac{-\sec^2 x}{-2\sin 2x} = \frac{\sec^2\frac{\pi}{4}}{2\sin\frac{\pi}{4}} = \frac{(\sqrt{2})^2}{2(1)} = \underline{\underline{1}}$$

(b) $$\lim_{x\to\frac{\pi}{6}}\frac{1-2\sin x}{\cos 3x}\ \underset{(\text{LHR})}{(0/0\text{ form})} = \lim_{x\to\frac{\pi}{6}}\frac{-2\cos x}{-3\sin 3x} = \frac{2\cos\frac{\pi}{6}}{3\sin\frac{\pi}{2}} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{3(1)} = \underline{\underline{\frac{\sqrt{3}}{3}}}$$

(c) $$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x}\ \underset{(\text{LHR})}{(\infty/\infty\text{ form})} = \lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{3\sec^2 3x} = \frac{1}{3}\left(\lim_{x\to\frac{\pi}{2}}\frac{\cos 3x}{\cos x}\right)^2 \underset{(\text{LHR})}{=} \frac{1}{3}\left(\lim_{x\to\frac{\pi}{2}}\frac{-3\sin 3x}{-\sin x}\right)^2 = \underline{\underline{3}}$$

(d) $$\lim_{x\to\pi}(\pi - x)\tan\frac{x}{2} = \lim_{y\to 0} y\tan\frac{\pi - y}{2} = \lim_{y\to 0} y\cot\frac{y}{2} = \lim_{y\to 0}\frac{y}{\tan\frac{y}{2}} \underset{(\text{LHR})}{=} \lim_{y\to 0}\frac{1}{\frac{1}{2}\sec^2\frac{y}{2}} = \underline{\underline{2}}$$

(e) $$L = \lim_{x\to 0}\left[\left(\frac{1}{x} - \frac{1}{\sin x}\right)^3 + 3\left(\frac{1}{x^2\sin x} - \frac{1}{x\sin^2 x}\right)\right]$$

$$= \left(\lim_{x\to 0}\frac{\sin x - x}{x\sin x}\right)^3 + 3\lim_{x\to 0}\frac{2\sin x - 2x}{x^2 - x^2\cos 2x} = (L_1)^3 + 3L_2 \qquad \text{(correspondingly)}$$

Apply LHR to both L_1 and L_2, $\quad L_1 \underset{(\text{LHR})}{=} \lim_{x\to 0}\dfrac{1-\cos x}{\sin x + x\cos x} \underset{(\text{LHR})}{=} \lim_{x\to 0}\dfrac{\sin x}{2\cos x - x\cos x} = \dfrac{0}{2(1)-0} = 0$

$$L_2 = \lim_{x\to 0}\frac{2\sin x - 2x}{x^2 - x^2\cos 2x} \underset{(\text{LHR})}{=} \lim_{x\to 0}\frac{\cos x - 1}{x - x\cos 2x + x^2\sin^2 x}$$

$$\underset{(\text{LHR})}{=} \lim_{x\to 0}\frac{-\sin x}{1 - \cos 2x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}$$

$$\underset{(\text{LHR})}{=} \lim_{x\to 0}\frac{-\cos x}{2\sin 2x + 4\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin 2x} = \frac{-1}{0} = \infty \qquad \therefore\quad L = \infty$$

2. **(f)** $L = \lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4} = \lim_{x\to 0}\frac{2\sin\frac{x+\sin x}{2}\sin\frac{x-\sin x}{2}}{x^4}$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\lim_{x\to 0}\frac{\sin\frac{x+\sin x}{2}}{\frac{x+\sin x}{2}}\lim_{x\to 0}\frac{\sin\frac{x-\sin x}{2}}{\frac{x-\sin x}{2}} = \frac{1}{2}L_1L_2L_3 \qquad \text{(correspondingly)}$$

$L_2 = L_3 = 1$ as $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

$$L_1 \underset{(LHR)}{=} \lim_{x\to 0}\frac{2x-2\sin x\cos x}{4x^3} = \lim_{x\to 0}\frac{2x-\sin 2x}{4x^3} \underset{(LHR)}{=} \lim_{x\to 0}\frac{2-2\cos 2x}{12x^2} \underset{(LHR)}{=} \lim_{x\to 0}\frac{4\sin 2x}{24x}$$

$$\underset{(LHR)}{=} \lim_{x\to 0}\frac{8\cos 2x}{24} = \frac{8}{24} = \underline{\underline{\frac{1}{3}}} \qquad \therefore \quad L = \frac{1}{2}\times\frac{1}{3}\times 1\times 1 = \underline{\underline{\frac{1}{6}}}$$

(g) $\lim_{x\to 0}\frac{x-x\cos x}{x-\sin x} \underset{(LHR)}{=} \lim_{x\to 0}\frac{1-\cos x+x\sin x}{1-\cos x} \underset{(LHR)}{=} \lim_{x\to 0}\frac{2\sin x+x\cos x}{\sin x} = \lim_{x\to 0}\left(2+\frac{x}{\sin x}\cos x\right) = \underline{\underline{3}}$

(h) $\lim_{x\to 0}\frac{\tan x - x}{x-\sin x} \underset{(LHR)}{=} \lim_{x\to 0}\frac{\sec^2 x-1}{1-\cos x} = \lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-1}{1-\cos x} = \lim_{x\to 0}\frac{1+\cos x}{\cos^2 x} = \frac{1+1}{1} = \underline{\underline{2}}$

3. **(a)** $\lim_{x\to 0}\frac{1-\cos x}{x^2} \underset{(LHR)}{=} \lim_{x\to 0}\frac{\sin x}{2x} = \underline{\underline{\frac{1}{2}}}$

(b) $\lim_{x\to 0}\frac{x-\tan x}{x^3} \underset{(LHR)}{=} \lim_{x\to 0}\frac{1-\sec^2 x}{3x^2} = -\lim_{x\to 0}\frac{\tan^2 x}{3x^2} = -\frac{1}{3}\left(\lim_{x\to 0}\frac{\tan x}{x}\right)^2 = \underline{\underline{-\frac{1}{3}}}$

(c) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x} \underset{(LHR)}{=} \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}}\cot x = \underline{\underline{0}}$

(d) $\lim_{x\to 1}(x^2-1)\tan\frac{\pi x}{2} = \lim_{x\to 1}\frac{x^2-1}{\cot\frac{\pi x}{2}} \underset{(LHR)}{=} \lim_{x\to 1}\frac{2x}{-\csc^2\frac{\pi x}{2}} = -\lim_{x\to 1}\left(2x\sin^2\frac{\pi x}{2}\right) = \underline{\underline{-2}}$

(e) $\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x} \underset{(LHR)}{=} \frac{\frac{1}{2\sqrt{4+x}}+\frac{1}{2\sqrt{4-x}}}{1} = \underline{\underline{\frac{1}{2}}}$

(f) $\lim_{x\to 3}\frac{\sqrt{3x}-\sqrt{12-x}}{2x-3\sqrt{19-5x}} \underset{(LHR)}{=} \lim_{x\to 3}\frac{\frac{3}{2\sqrt{3x}}+\frac{1}{2\sqrt{12-x}}}{2+\frac{15}{2\sqrt{19-5x}}} = \frac{\frac{1}{2}+\frac{1}{6}}{2+\frac{15}{4}} = \underline{\underline{\frac{8}{69}}}$

(g) $\lim_{x\to\infty}\left(x^2-\frac{x}{\tan\frac{1}{x}}\right) = \lim_{y\to 0}\left(\frac{1}{y^2}-\frac{1}{y\tan y}\right) = \lim_{y\to 0}\left(\frac{1}{y^2}-\frac{\cos y}{y\sin y}\right) = \lim_{y\to 0}\left(\frac{\sin y - y\cos y}{y^2\sin y}\right)$

$$\underset{(LHR)}{=} \lim_{y\to 0}\frac{\cos y-\cos y+y\sin y}{y^2\cos y+2y\sin y} = \lim_{y\to 0}\frac{\sin y}{y\cos y+2\sin y} \underset{(LHR)}{=} \frac{\cos y}{\cos y - y\sin y+2\cos y} = \underline{\underline{\frac{1}{3}}}$$

$\therefore$ Result follows by replacing x by positive integral of n .

4. **(a)** $\lim_{x\to 0}\frac{x+\tan x}{\sin 2x} \underset{(LHR)}{=} \lim_{x\to 0}\frac{1+\sec^2 x}{2\cos 2x} = \frac{1+1}{2(1)} = \underline{\underline{1}}$

(b) $\lim_{x\to 0}\frac{\tan^{-1} x}{x} = \lim_{y\to 0}\frac{y}{\tan y} = \underline{\underline{1}}$

(c) $\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+x)} \underset{(LHR)}{=} \lim_{x\to 0}\frac{2e^{2x}}{1/(1+x)} = \frac{2}{1/1} = \underline{\underline{2}}$

(d) $\lim_{x\to 0}\frac{a^x-1}{x} \underset{(LHR)}{=} \lim_{x\to 0}\frac{a^x \ln a}{1} = \underline{\underline{\ln a}}$

(e) $\lim_{x\to\infty} x^3 e^{-x} \underset{(LHR)}{=} \frac{3x^2}{e^x} \underset{(LHR)}{=} \lim_{x\to\infty}\frac{6x}{e^x} \underset{(LHR)}{=} \lim_{x\to\infty}\frac{6}{e^x} = \underline{\underline{0}}$

(f) $\lim_{x\to 0^+}\sin x \ln x = \lim_{x\to 0^+}\frac{\ln x}{\csc x} \underset{(LHR)}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc x \cot x} = -\lim_{x\to 0^+}\frac{\sin x \tan x}{x} \underset{(LHR)}{=} -\lim_{x\to 0^+}\left(\sin x \sec^2 x + \cos x \tan x\right) = \underline{\underline{0}}$

(g) $\lim_{x\to 0}\left(\frac{1}{x} - \csc x\right) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$, see **2(e)**

(h) $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\left(\frac{x\ln x - x + 1}{(x-1)\ln x}\right) \underset{(LHR)}{=} \lim_{x\to 1}\frac{x(1/x)+\ln x - 1}{(x-1)(1/x)+\ln x} = \lim_{x\to 1}\frac{x\ln x}{x-1+x\ln x}$

$\underset{(LHR)}{=} \lim_{x\to 1}\frac{x(1/x)+\ln x}{1+x(1/x)+\ln x} = \lim_{x\to 1}\frac{1+\ln x}{2+\ln x} = \frac{1}{\underline{\underline{2}}}$

(i) $\lim_{x\to\frac{\pi}{2}}(\tan 5x - \tan x) = \lim_{x\to\frac{\pi}{2}}\left(\frac{\sin 5x}{\cos 5x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}}\frac{\sin 5x \cos x - \cos 5x \sin x}{\cos 5x \cos x} = \lim_{x\to\frac{\pi}{2}}\frac{\sin 4x}{\cos 5x \cos x}$

$\underset{(LHR)}{=} \lim_{x\to\frac{\pi}{2}}\frac{4\cos 4x}{-\cos 5x \sin x - 5\sin 5x \cos x} = \frac{4(1)}{-0-0} = \underline{\underline{\infty}}$

(j) $\lim_{x\to\infty}\frac{\ln(e^x+x^2)}{x^2} \underset{(LHR)}{=} \lim_{x\to\infty}\frac{\frac{1}{e^x+x^2}(e^x+2x)}{2x} = \lim_{x\to\infty}\frac{e^x+2x}{2x(e^x+x^2)} \underset{(LHR)}{=} \lim_{x\to\infty}\frac{e^x+2}{2(e^x+x^2)+2x(e^x+2x)}$

$= \frac{1}{2}\lim_{x\to\infty}\frac{e^x+2}{e^x+3x^2+xe^x} \underset{(LHR)}{=} \frac{1}{2}\lim_{x\to\infty}\frac{e^x}{e^x+6x+e^x+xe^x} = \frac{1}{2}\lim_{x\to\infty}\frac{e^x}{2e^x+6x+xe^x}$

$\underset{(LHR)}{=} \frac{1}{2}\lim_{x\to\infty}\frac{e^x}{2e^x+6+e^x+xe^x} = \frac{1}{2}\lim_{x\to\infty}\frac{e^x}{3e^x+6+xe^x} \underset{(LHR)}{=} \frac{1}{2}\lim_{x\to\infty}\frac{e^x}{3e^x+e^x+xe^x} = \frac{1}{2}\lim_{x\to\infty}\frac{1}{4+x} = \underline{\underline{0}}$

(k) $y = \lim_{x\to\infty} x^{\frac{1}{x}} \quad \Rightarrow \ln y = \lim_{x\to\infty}\frac{\ln x}{x} \underset{(LHR)}{=} \lim_{x\to\infty}\frac{1/x}{1} = \lim_{x\to\infty}\frac{1}{x} = 0 \Rightarrow y = e^0 = \underline{\underline{1}}$

(l) $y = \lim_{x\to 0} x^{\sin x} \quad \Rightarrow \ln y = \lim_{x\to 0}\frac{\ln x}{1/\sin x} = \lim_{x\to 0}\frac{\ln x}{\csc x} \underset{(LHR)}{=} \lim_{x\to 0}\frac{1/x}{-\csc x \cot x} = -\lim_{x\to 0}\frac{\sin x \tan x}{x}$

$\underset{(LHR)}{=} -\lim_{x\to 0}\frac{\sin x \sec^2 x + \tan x \cos x}{1} = 0 \quad \Rightarrow y = e^0 = \underline{\underline{1}}$

(m) $y = \lim_{x\to 0^+} x^x \quad \Rightarrow \ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \underset{(LHR)}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = -\lim_{x\to 0^+} x \quad \Rightarrow y = e^0 = \underline{\underline{1}}$

(n) $y = \lim_{x\to 1} x^{\frac{1}{1-x}} \quad \Rightarrow \ln y = \lim_{x\to 1}\frac{\ln x}{1-x} \underset{(LHR)}{=} \lim_{x\to 1}\frac{1/x}{-1} = -\lim_{x\to 1}\frac{1}{x} = -1 \quad \Rightarrow y = e^{-1} = \frac{1}{\underline{\underline{e}}}$